H-55026 Seat No.________

M. Sc. (Part - I) Examination

April / May – 2003

Statistics : Paper - II

(Probability Theory & Distributions)

Time : **3** Hours] [Total Marks : **75**

Instructions : (1) All questions carry **equal** marks.
(2) Use of calculator and statistical tables is permissible.

1 (a) Define a probability space and show that probability measure P defined on the space is countably subadditive.

(b) Define a distribution function of random variable and show that set of discontinuity points of a distribution function is atmost countable.

(c) State the decomposition theorem for the distribution functions.

A distribution function is given by

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x^2}{3} \text{ if } 0 \le x < 1 \\ \frac{1}{3} + \left(\frac{x-1}{2}\right)^2 \text{ if } 1 \le x < 2 \\ 1 \text{ if } x \ge 2. \end{cases}$$

Obtain the decomposition of F into its discrete and continuous parts.

OR

1 (a) State and prove Markov's inequality. Derive Chebyshev's inequality from Markov's inequality.

(b) If $E(X_1) = E(X_2) = 0$, $V(X_1) = V(X_2) = 1$ and $COV(X_1, X_2) = \rho$ then show that $E\left[\max\left(X_1^2, X_2^2\right)\right] \le 1 + \sqrt{1-\rho^2}$. Hence derive the Berge inequality for correlated random variables.

(c) Show that convergence almost surely implies convergence in probability.

2 (a) Define characteristic function of random variable. State and prove inversion theorem on characteristic function.

(b) State Borel Cantelli Lemma. If $\{A_n\}$ is a sequence of events such that $P(A_n)=\frac{1}{2}P(A_{n-1})$ and A_1 is a certain event then find $P\ (lim\ Sup\ A_n)$.

(c) State and prove Helly-Bray theorem.

OR

2 (a) Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E(X_n)=\mu<\infty$. Then show that $\overline{X}_n \xrightarrow{P} \mu$.

(b) State and prove Liapounov's form of central limit theorem.

(c) If $\{X_n\}$ is a sequence of independent random variables with the distributions : $\boldsymbol{P\left[X_n=\pm n^{\lambda}\right]=\frac{1}{2}}$ where λ is constant. Examine whether the central limit theorem holds for any λ.

3 (a) Define the terms :

(i) Markov Chain

(ii) Absorbing State

(iii) Stationary distribution.

(b) If $P(X_0=i)=\frac{1}{3}$ for i=1,2,3 and

$P=\begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$, then compute $\boldsymbol{P\left(X_2=i\right)}$ for $i=1,2,3$.

(c) Stating the postulates of a pure birth process, obtain the difference-differential equations governing,

$$P_n(t)=P\left(x(t)=n/X(o)=i\right)$$

OR

3 (a) Define a contagious distribution and obtain its probability generating function. Hence or otherwise derive the probability function of Poisson-Pascal distribution.

(b) Let $X_1, X_2, \ldots\ldots, X_N$ be independent random variables with common distribution F and let N be a random variable independent of X_j. Define $Y=\sum_{i=1}^{n} X_i$ and let $\phi_y(t)$ be the characteristics function (ch.f) of Y. $\phi_1(t)$ and $\phi_2(t)$ are the ch.fs of random variables N and X respectively then show that $\phi_Y(t) = \phi_1\left[-i\log\left(\phi_2(t)\right)\right]$.

(c) Explain the method to obtain ML estimates of the parameters of Neyman type-A distribution.

4 (a) Define Poisson-Binomial distribution and derive its r^th^ factorial cumulant. Hence obtain moment estimators of the parameters of the distribution.

(b) Suppose the number of automobile accidents in a certain area in a week is a random variable with mean μ_1 and variance σ_1^2. The number of persons in sured in each accident is independent from accident to accident and has mean μ_2 and variance σ_2^2. Let W denote the number of persons injured in automobile accidents in the area in a week. Find $E(W)$ and $V(W)$.

(c) Define multinomial distribution $M_k\,(n, p_1, p_2, \ldots\ldots, p_k)$ for the distribution show that

$$\rho_{12.34\ldots m} = -\left\{\frac{p_1 p_2}{(1-p_1-p_3-\ldots\ldots-p_m)(1-p_2-p_3-\ldots.\ p_m)}\right\}^{1/2}$$ where $m<k$.

OR

4 (a) Define multivariate normal distribution. Let $\underset{\sim}{x} \sim N_n\left(\underset{\sim}{\mu}, \Sigma\right)$ then show that $\underset{\sim}{y} = L\underset{\sim}{x} \sim N_p\left(L\underset{\sim}{\mu}, L\Sigma L'\right)$ where L is a known matrix of order $n\times p$. Hence show that $\underset{\sim}{y}_1 = L_1\underset{\sim}{x}$ and $\underset{\sim}{y}_2 = L_2\underset{\sim}{x}$ are independently distributed as normal iff $L_1 \Sigma L_2^1 = 0$.

(b) Let $x \sim N_p\left(\underset{\sim}{\mu}, \Sigma\right)$. Derive the distribution of $\underset{\sim}{X}' A \underset{\sim}{X}$, where A is positive definite real symmetric matrix. Hence show that $E\left(\underset{\sim}{X}' A \underset{\sim}{X}\right) = tr(A\Sigma) + \underset{\sim}{\mu}' A\mu$.

(c) Let $X_n = \text{maxi}\ (X_1, X_2 \ldots., X_n)$. Show that $E\left(X_{(n)}\right) = E\left(X_{(n-1)}\right) + \int_0^\infty F^{n-1}(x)[1-F(x)]dx, n = 2, 3, \ldots.$. Also find $E\left(X_{(n)}\right)$ if X_i`s have the common distribution function $F(x) = 1 - e^{-\beta x},\ x \geq 0,\ \beta > 0$.

5 (a) Let $X_1, X_2, ..., X_n$ be iid $N(0,1)$ random variables. Define $Y = \sum_{i=x}^{n} (X_i + \lambda)^2$. Obtain the sampling distribution of Y. Also find the r^{th} cumulant of Y. What will be the distribution of Y when $\lambda = 0$?

(b) Define noncentral F and doubly noncentral F variates. Derive the distribution of noncentral F variate and find its mean.

OR

5 (a) If X follows $N(\mu, 1)$ distribution and Y is an independent chi-square variate with n degrees of freedom then obtain the distribution of $\frac{\sqrt{n}X}{\sqrt{Y}}$. Give some applications of the distribution.

(b) Prove re-productive property of noncentral chi-square distribution. If X and Y are two independent random variables such that X is chi-square variate with r df and Y is noncentral chi-square with 1 df and λ non-centrality parameter then obtain the distribution of $X+Y$. Find its mean and variance.