55011 Seat No. ______

M. A. / M. Sc. (Part - I) Examination

April / May – 2003

Statistics, Matrix Algebra & Measure Theory : Paper - I

Time : **3** Hours] [Total Marks : **75**

Instructions : (1) All questions carry **equal** marks.
(2) Use of calculator and statistical tables is permissible.

1 (a) Let A, B, C, D be $n \times n$ matrices, where $|D| \neq 0$ and D and C commutes. Show that :

(1) $\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = |A| \cdot |D|$

(2) $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - BC|$.

(b) Show that $tr\ AB = tr\ BA$, for any two matrices A and B as long as the product exist. Also show that $tr\,(AB)^2 = tr(BA)^2$.

OR

1 (a) Let A be $m \times n$ and B be $n \times p$ matrices. If $\rho(A) = n$, show that $\rho(AB) = \rho(B)$. Also show that, if $\rho(B) = n$, $\rho(AB) = \rho(A)$.

(b) **f** $A : q \times m$, $B : m \times n$ and $C : n \times p$, show that $\rho(AB) + \rho(BC) - \rho(B) \leq \rho(ABC) \leq \min\ \{\rho(AB),\ \rho(BC)\}$.

2 (a) Let A be nonsingular matrix of order n and let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two n–component columon vectors such that $\left(1 + \underset{\sim}{v}\, A^{-1} \underset{\sim}{u}\right)$ is non–zero. Then show that $\left(A + \underset{\sim}{u}\ \underset{\sim}{v}'\right)^{-1} = A^{-1} - \dfrac{A^{-1} \underset{\sim}{u}\ \underset{\sim}{v}'\, A^{-1}}{1 + \underset{\sim}{v}'\ A^{-1} \underset{\sim}{u}}$.

(b) If $A : n \times n$ and $B : m \times m$ are non–singular matrices and $C : m \times n$, show that

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -B^{-1} \subset A^{-1} & B^{-1} \end{pmatrix}$$

(c) Let A^{-} be a g–inverse of A and B^{-} be a g–inverse of B. Prove that $B^{-}A^{-}$ is a g–inverse of AB if and only if $A^{-}ABB^{-}$ is idempotent.

OR

2 (a) Define the Moore–Penrose g–inverse of a matrix. If $A^{\dagger}$ denotes the Moore–Penrose g–inverse of A, show that :

(1) $(A')^{\dagger} = (A^{\dagger})'$

(2) $(A' \ A)^{\dagger} = A^{+}(A')^{+}$

(b) If A is a real symmetric matrix of order n, show that there exists and orthogonal matrix C such that $C' AC = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are characteristics roots of A.

(c) Show that the number of non–zero characteristics roots of a matrix cannot exceed its rank.

3 (a) Let $A_1, A_2, ..., A_k$ be square matrices of order n and $A = \sum_{i=1}^{k} A_i$. If A is an idempotent matrix and $\rho(A) = \sum_{i=1}^{k} \rho(A_i)$, show that each $A_i \ (i = 1, 2, ..., k)$ is idempotent and $A_i A_j = 0$ for $i, \ j \in \{1, 2, ..., k\}; \quad i \neq j$.

(b) If $\lambda_1 > \lambda_2 > ...... > \lambda_n$ are characteristic roots of a real symmetric matrix A of order $n \times n$ show that

$$\lambda_n \leq \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} \leq \lambda_1 \qquad \text{for} \quad \underset{\sim}{x} \neq \underset{\sim}{0}.$$

(c) If $\lambda_1, \lambda_2, ..., \lambda_n$ are the characteristic roots of a matrix A

of order n, show that (1) $tr\ A = \sum_{i=1}^{n} \lambda_i$ (2) $|A| = \prod_{i=1}^{n} \lambda_i$

OR

3 (a) Define the following terms :

(1) Hahm and Jordan decomposition

(2) Lebesgue decomposition

(3) Equivalence of two measures

(4) Singularity of two measures

(5) Outer measurable set.

(b) Show that every set of outer measure zero is μ^*–measurable set.

(c) Give an example of a set function which is measure. Define outer measure and construct on outer measure of measure which you define. Why we define outer measure ?

4 (a) Define the terms :

(1) σ–ring

(2) σ–field

(3) monotone class

(4) Hereditary class.

(b) Give an example of a class which is a hereditary class but not a σ–field.

(c) Show that every union of arbitrary sets in a σ–field can be represented as a union of mutually disjoint sets in the same σ–field.

(d) If μ is a measure on field $\Im$ and $\{A_n\}$ is a sequence of sets in $\Im$ such that $\lim\ A_n \in \Im$, $\mu(A_n) < \infty$ for at least one n then show that $\mu(\lim\ A_n) = \lim \mu(A_n)$.

OR

4 (a) State the extension theorem on measure and use it to define Lebesgue–Stieltjes measure.

(b) If μ_F is the Lebesgue–Stieltjes measure with respect to a Stieltjes measurable function $F : R \to R$ show, in usual notations, that :

(1) $\mu_F(\{a\}) = F(a+0) - F(a-0)$ for $a \in R$

(2) $\mu_F((a, b)) = F(b-0) - F(a)$ for $-\infty < a \le b < \infty$.

If F is a distribution function given by

$$F(x)=\begin{cases}0 & \text{if} & x<0\\ \frac{1}{4} & \text{if} & 0\le x<1\\ 1 & \text{if} & x\ge 1\end{cases}$$

Find $\mu_F=(0,\ 1)$ and $\mu_F\,[0,\ 1]$.

(c) State and prove "Continuity theorem on measure".

5 (a) Define the integral of a simple measurable function over a measure space $(x,\quad,\mu)$. If f is a simple measurable function such that $f\ge 0$ almost everywhere then show that $\int f\,d\mu\ge 0$.

(b) IF λ is the Lebesgue measure in R find $\int fd\lambda$ where f is defined by

$$f=\begin{cases}-2 & \text{for} & -1<x<0\\ 1 & \text{for} & x=0\\ 2 & \text{for} & 0<x\le 1\\ 3 & \text{otherwise} & \end{cases}$$

What is the value of $\int_A f\,d\lambda$ where $A=(3,\ 10)$?

(c) Define measurable function. Show that random variable is an example of measurable function.

OR

5 (a) State and prove "Monotone Convergence Theorem".

(b) Let $\{f_n\}$ be a sequence of non–negative measurable functions in a measure space $(x,\quad,\mu)$, show that :

(1) $\int\left(\sum_{n=1}^{\infty} f_n\right)d\mu=\sum_{n=1}^{\infty}\int f_n\,d\mu$

(2) $\int(\liminf f_n)\,d\mu\le\liminf\int f_n\,d\mu$.

(c) Show that a simple function $f(x)=\sum_{i=1}^{n} C_i\,\chi_{E_i}(x)$ is integrable iff $c_i=0$ for each integer i such that $\mu(E_i)=\infty$